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Infinite-dimensional Lie groups and algebraic geometry in soliton theory

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We study several methods of describing 'explicit' solutions to equations of Korteweg–de Vries type: (i) the method of algebraic geometry (Krichever, I.M. *Usp. mat. Nauk* **32**, 183–208 (1977)); (ii) the Grassmannian formalism of the Kyoto school (iii) acting on the trivial solution by the 'group of dressing transformations' (Zakharov, V. E. & Shabat, A. B. *Funct. Anal. Appl.* **13** (3), 13–22 (1979)). I show that the three methods are more or less equivalent, and in particular that the 'τ-functions' of method (ii) arise very naturally in the context of method (iii).

1. Introduction

In this lecture we shall be concerned with the Korteweg–de Vries (K.d.V.) equation

$$u_t = u_{xxx} + 6uu_x, \tag{1.1}$$

the modified Korteweg–de Vries (m.K.d.V.) equation

$$q_t = q_{xxx} - 6q^2 q_x, \tag{1.2}$$

and the Miura transformation

$$u = q_x - q^2 \tag{1.3}$$

relating them (if $q(x, t)$ is any solution of the m.K.d.V. equation then the function $u(x, t)$ defined by (1.3) is a solution of the K.d.V. equation). We shall concentrate on just one aspect of the elaborate theory of these equations: they have a fairly large class of explicit solutions, which are most elegantly (and naturally?) written in the form

$$q = \partial/\partial x \log \tau_0/\tau_1 \tag{1.4}$$

or

$$u = 2\partial^2/\partial x^2 \log \tau. \tag{1.5}$$

In the theory of Krichever (1977) the 'τ-functions' occurring in these formulae are essentially θ-functions of hyperelliptic Riemann surfaces: the class of solutions that we shall discuss is larger than Krichever's, but shares with it the property that the τ-functions are entire functions of the variables x and t.

The material is organized as follows. In §2 I introduce (for the K.d.V. equation) the class of solutions just mentioned, and explain the connection with the 'group of hidden symmetries' G of the equation: G is the group of maps from the unit circle S^1 to $\mathrm{SL}(2, \mathbb{C})$. The exposition follows Segal & Wilson (1985), which in turn follows Krichever (1977) and Date *et al.* (1981). In §3 I discuss (for the m.K.d.V. equation) the 'dressing' construction of Zakharov & Shabat (1979): it amounts to defining an action of G on a certain space associated with the equation.

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The G-orbit of the trivial solution $q \equiv 0$ covers (via the Miura transformation) the class of solutions of the K.d.V. equation considered in §2. Section 4 shows how the τ-functions, and in particular the formula (1.4), arise in the context of the Zakharov–Shabat construction: the key point is that we have to bring into play the universal central extension of G. The exposition in §§3, 4 follows Wilson (1984).

According to Drinfel'd & Sokolov (1981, 1984), the K.d.V. and m.K.d.V. equations have natural generalizations in which the symmetry group G is the group corresponding to any affine Kac–Moody algebra (see Kac 1983). One of the motivations for the work presented here was to understand how to implement the apparently new ideas of Date *et al.* (1981) in the more general context of the affine algebras. Essentially the entire theory goes through in this context; although I have not tried to discuss this here, from §3 onwards the material is presented in such a way that the generalization is immediate. Nevertheless, a few things do not survive in the general case; one example of such a casualty is the very popular formula (1.5). The reason is explained briefly at the end of §4.

2. The curve–Grassmannian construction for the K.d.V. equation

We first review quickly the Lax representation of the K.d.V. equation. Set

$$\mathrm{L} = \partial^2 + u(x,t),$$

where $\partial \equiv \partial/\partial x$. Then the K.d.V. equation (1.1) is equivalent to

$$\mathrm{L}_t = [\mathrm{P}_3, \mathrm{L}], \tag{2.1}$$

where
$$\tfrac{1}{4}\mathrm{P}_3 = \partial^3 + \tfrac{3}{2}u\partial + \tfrac{3}{4}u_x.$$

The form of the operator P_3 is determined by the requirement that the right side of (2.1) should be a multiplication operator (i.e. of order zero in ∂), together with a homogeneity property: P_3 is homogeneous of weight 3 if we give ∂ weight 1, and $u^{(j)} = \partial^j u/\partial x^j$ weight $2+j$. More generally, for each integer $r \geqslant 0$ there is a unique operator P_{2r+1} of the form

$$\mathrm{P}_{2r+1} = \partial^{2r+1} + (\text{lower terms})$$

with coefficients that are homogeneous polynomials in u and its x-derivatives $u^{(j)}$, and such that $[\mathrm{P}_{2r+1}, \mathrm{L}]$ is a multiplication operator. The equation

$$\mathrm{L}_t = [\mathrm{P}_{2r+1}, \mathrm{L}]$$

is then equivalent to a single evolution equation for u, which is called the rth *equation of the K.d.V. hierarchy*. A basic fact is that if we regard each of these equations as defining a flow on a suitable space of functions $u(x)$, then these flows all commute with each other. It is therefore reasonable to consider simultaneous solutions of any number of equations of the hierarchy; that is, functions $u(x, t_3, t_5, \ldots)$ such that if we fix the values of all the t_i except one, say t_{2r+1}, then the resulting function of x and t_{2r+1} is a solution of the rth equation of the hierarchy.

Now we describe Krichever's construction, in a slightly modified form taken from Segal & Wilson (1985). In $\mathbb{C}^2$ we consider the algebraic curve X' with equation

$$\mu^2 = f(\lambda),$$

where f is a polynomial of degree $2g+1$. The projection onto the λ-plane exhibits X' as a double covering of $\mathbb{C}$; by adding a single point at infinity, say x_∞, we get a compact curve X, which

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is a double covering of the Riemann sphere S^2. If all the roots of the polynomial f are distinct, then X is non-singular, that is, it is a compact hyperelliptic Riemann surface. Near the point x_∞ we fix a local parameter z^{-1} on X so that the projection $X \to S^2$ is given (locally) by $z \to z^2$; we suppose the validity of this parameter extends as far as the unit circle, so that we can use it to identify a small neighbourhood D of x_∞ in X with the disc $\{z : |z| \geqslant 1\}$; in particular the boundary of this neighbourhood gets identified with the unit circle $S^1 \subset S^2$. Fix g points $P_1, \ldots, P_g$ in $X \backslash D$, and consider the following space W of functions on S^1: *W consists of all square-integrable functions $\varphi: S^1 \to \mathbb{C}$ such that φ is the boundary value of a function on $X \backslash D$ which is holomorphic except for (possible) simple poles at the points $P_1, \ldots, P_g$.* Thus W is a closed subspace of the Hilbert space $H = L^2(S^1, \mathbb{C})$. All the information that we shall need from the data $(X, D, P_1, \ldots, P_g)$ is already contained in the single object W.

Example 1. Suppose X is just the Riemann sphere. Here $g = 0$, so there are no points P_i. Thus in this case W is simply the space H_+ of boundary values of functions holomorphic in the disc $\{z : |z| < 1\}$, that is, functions whose Fourier series involve only non-negative powers of z.

Now let us return to the general case.

PROPOSITION 1. (i) *There is a unique function $\psi_W(x, \boldsymbol{t}; z)$ of the form*

$$\psi_W = \exp\,(xz + t_3 z^3 + \ldots)\,(1 + O(z^{-1}))$$

such that $\psi_W(x, \boldsymbol{t}; .)$ belongs to W for each fixed value of $(x, \boldsymbol{t})$.

(ii) *There are unique ordinary differential operators* L, P_{2r+1} *of the form*

$$\mathrm{L} = \partial^2 + u(x, \boldsymbol{t}), \quad \mathrm{P}_{2r+1} = \partial^{2r+1} + \ldots$$

such that

$$\left.\begin{aligned} \mathrm{L}\psi_W &= z^2 \psi_W, \\ \partial\psi_W/\partial t_{2r+1} &= \mathrm{P}_{2r+1}\,\psi_W. \end{aligned}\right\} \tag{2.2}$$

(iii) *The function $u(x, \boldsymbol{t})$ is a solution of the K.d.V. equations.*

Here and later $\boldsymbol{t}$ denotes $(t_3, t_5, \ldots)$. The function ψ_W is called the *Baker function* of W (or of the algebro-geometric data from which W was constructed). Of course, part (iii) of the proposition follows at once from part (ii), since by definition the K.d.V. equations are the integrability condition for the system (2.2).

It is worth noting that the g-soliton solutions of the K.d.V. equations can be obtained as a special case of the above construction: we take X to be a rational curve with g double points, i.e. X' is given by the equation

$$\mu^2 = \lambda \prod_1^g (\lambda - a_i)^2.$$

As a still more special case, if we take X' to be the curve $\mu^2 = \lambda^{2g+1}$, we get the rational solutions to the K.d.V. equations studied by (among others) Adler & Moser (1978).

Now, all the subspaces $W \subset H$ that we have constructed have the following two properties:

(*a*) $z^2 W \subset W$;

(*b*) the orthogonal projection $W \to H_+$ is a Fredholm operator of index zero (that is, its kernel and cokernel have the same finite dimension).

Indeed, (*a*) is obvious: by construction, W is invariant under multiplication by any function on S^1 that extends to a holomorphic function on $X \backslash D$, and by hypothesis, z^2 has that property. Property (*b*) is not surprising either: in general we have made only a 'compact perturbation' of the special case considered in example 1, so we expect W to be 'close' to the subspace H_+

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arising in that case. A proof of (*b*) can be found in Segal & Wilson (1985); the proof shows that if the points $P_1, \ldots, P_g$ are in general position, the projection $W \to H_+$ is even an isomorphism. Let us denote by $Gr^{(2)}$ the Grassmannian of all closed subspaces W of H satisfying the conditions (*a*) and (*b*). Not every point $W \in Gr^{(2)}$ arises from an algebraic curve in the way we have described. Nevertheless, if we examine Krichever's (1977) proof of proposition 1, we see that it uses only the two properties (*a*) and (*b*); that is proposition 1 remains true for any $W \in Gr^{(2)}$. Thus we have defined a map

$$Gr^{(2)} \to \{\text{solutions of the K.d.V. equations}\}.$$

It is important to note that the solutions $u(x, \boldsymbol{t})$ that we have obtained are of a very special nature: for example, for reasons that will become clear later, they are all meromorphic functions defined on the entire complex planes of the variables $x, t_3, \ldots$.

We are now in a position to see how loop groups enter into the picture. Let G be the infinite dimensional Lie group of all smooth maps from the circle S^1 to $\mathrm{SL}(2, \mathbb{C})$. Let $H^{(2)}$ be the space of all row vectors (f_0, f_1) with $f_i \in H = L^2(S^1, \mathbb{C})$; i.e. $H^{(2)}$ is the Hilbert space $L^2(S^1, \mathbb{C}^2)$. The group G acts on $H^{(2)}$ in an obvious way:

$$g \circ (f_0, f_1) = (f_0, f_1)\, g^{-1}$$

(the juxtaposition on the right is just the usual product of a vector and a matrix). On the other hand, there is a natural isomorphism $H^{(2)} \cong H$ given by $(f_0, f_1) \leftrightarrow f$, where

$$f(z) = f_0(z^2) + z f_1(z^2).$$

We use this isomorphism to transfer the G-action to H.

Proposition 2. *The Grassmannian $Gr^{(2)}$ is essentially the G-orbit of H_+ under this action.*

Here the word 'essentially' refers to two things: to make the proposition literally true we should need to take G to be the group of loops in $\mathrm{GL}(2, \mathbb{C})$, rather than $\mathrm{SL}(2, \mathbb{C})$, and we should also have to allow a wider class of loops than just the smooth ones. Neither of these points is of any importance for our purposes: from now on we shall redefine $Gr^{(2)}$ to be the G-orbit of H_+.

Proposition 2 means that $Gr^{(2)}$ is a homogeneous space of G: that is, we have

$$Gr^{(2)} \cong G/P,$$

where $P \subset G$ is the isotropy group of H_+. It is easy to describe P explicitly: it is the subgroup of G consisting of those functions on the circle that are boundary values of holomorphic maps $D_0 \to \mathrm{SL}(2, \mathbb{C})$, where D_0 is the disc $\{z : |z| < 1\}$.

In this section we have seen how to assign a (very special) solution of the K.d.V. equations to each point of the homogeneous space G/P. It is natural to ask whether this is not just part of a much bigger picture, i.e. whether our G/P is not just one orbit of a G-action on a larger space that maps onto a larger space of solutions of the K.d.V. equations. This is indeed the case; however, to understand it we have to start by considering the m.K.d.V. equations (and then regard the K.d.V. equations as quotient systems of these, via the Miura transformation).

3. The dressing action for the m.K.d.V. equation

As we have seen, the K.d.V. equations are associated with the eigenvalue problem $\mathrm{L}\psi = z^2\psi$, where $\mathrm{L} = \partial^2 + u$. On the other hand, the Miura transformation (1.3) can be written in the form

$$\mathrm{L} = (\partial - q)(\partial + q).$$

That suggests that the m.K.d.V. equations should be associated with the first-order system

$$(\partial+q)\,\psi_0 = z\psi_1,$$
$$(\partial-q)\,\psi_1 = z\psi_0$$

(which clearly implies $L\psi_0 = z^2\psi_0$). We can write this system in the form

$$\Psi_x = U\Psi \tag{3.1}$$

where $\Psi = (\psi_0, \psi_1)^t$

and
$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} z - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} q. \tag{3.2}$$

I shall explain the connection of the m.K.d.V. equations with the system (3.1) in more detail in a moment; but first we must digress to clear up a potentially confusing point concerning the symmetry condition satisfied by the matrix U.

Let $\mathfrak{g}$ be the Lie algebra of smooth maps $f: S^1 \to \mathfrak{sl}(2, \mathbb{C})$ that satisfy the following condition: in the Fourier expansion of f the coefficients of the even powers of z are diagonal matrices (of trace zero), while the coefficients of the odd powers of z have zero diagonal entries. The matrix U in (3.2) clearly satisfies this condition. It is equivalent to say that $\mathfrak{g}$ consists of the smooth maps that have the form

$$z \to \begin{bmatrix} a(z^2) & zb(z^2) \\ z^{-1}c(z^2) & d(z^2) \end{bmatrix} \tag{3.3}$$

(with $d = -a$). Let G be the group corresponding to the Lie algebra $\mathfrak{g}$, i.e. the group of all smooth maps $S^1 \to \mathrm{SL}(2, \mathbb{C})$ that have the form (3.3). In the previous section G denoted the group of *all* maps from S^1 to $\mathrm{SL}(2, \mathbb{C})$ (not necessarily of the form (3.3)). However, there is a natural isomorphism between these two groups G, namely

$$\begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix} \leftrightarrow \begin{bmatrix} a(z^2) & zb(z^2) \\ z^{-1}c(z^2) & d(z^2) \end{bmatrix}. \tag{3.4}$$

In what follows we shall use this isomorphism to identify the two groups G, that is, we shall regard them as two different concrete realizations of the same underlying abstract group. The realization used in §2 (with no symmetry condition on the maps) we shall call the *standard realization* of G; the realization that we have just introduced (as maps satisfying (3.3)) will be called the *principal realization* of G. For the rest of this section we shall work with the principal realization.

Next, I explain the so-called zero curvature representation of the m.K.d.V. equations. By definition, the rth equation of the m.K.d.V. hierarchy is the integrability condition for the system

$$\left.\begin{aligned} \partial\Psi/\partial x &= U\Psi, \\ \partial\Psi/\partial t_{2r+1} &= V_r\,\Psi, \end{aligned}\right\} \tag{3.5}$$

where U is given by (3.2) and V_r is a polynomial of degree $2r+1$ in z, which is essentially determined by the requirement that the integrability condition for (3.5), namely

$$[\partial - U,\ \partial/\partial t_{2r+1} - V_r] = 0, \tag{3.6}$$

or equivalently

$$\partial U/\partial t_{2r+1} = V_x - [U, V_r], \tag{3.7}$$

should reduce to a single evolution equation for q. More precisely, for each $r \geqslant 0$ there is a unique matrix

$$V_r = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} z^{2r+1} + \sum_0^{2r} v_i z^i$$

(of the form (3.3)) such that (i) the right side of (3.7) is of degree zero in z (ii) the entries in the coefficients v_i are polynomials in q and its x-derivatives $q^{(j)}$, and V_r is homogeneous of weight $2r+1$ if we give z weight 1 and $q^{(j)}$ weight $j+1$.

For example, we have

$$V_1 = \begin{bmatrix} -qz^2 + \frac{1}{4}(2q^3 - q_{xx}) & z^3 + \frac{1}{2}(q_x - q^2)\, z \\ z^3 - \frac{1}{2}(q_x + q^2)\, z & qz^2 - \frac{1}{4}(2q^3 - q_{xx}) \end{bmatrix};$$

the corresponding equation (3.7) is (apart from a factor 4) equivalent to the m.K.d.V. equation (1.2) for q. Just as for the K.d.V. equations, the flows on the space of functions $q(x)$ defined by (3.7) for different values of r all commute with each other, so we can consider simultaneous solutions $q(x, t_3, t_5, \ldots)$ to any number of equations of the hierarchy.

We now focus attention on the function Ψ occurring in (3.5). We shall interpret it as a function of $(x, \boldsymbol{t})$ with values in the loop group G (i.e. as a 'fundamental solution matrix', rather than as a single solution of the system). Following Zakharov & Shabat (1979), we consider the 'gauge symmetries' of the system, that is, we consider the possibility of replacing Ψ by $\chi(x, \boldsymbol{t}; z)\, \Psi$, where χ is some other function of $(x, \boldsymbol{t})$ with values in G. Under such a transformation U gets replaced by $\chi U \chi^{-1} + \chi_x \chi^{-1}$ and V_r by a similar expression. Of course, for any choice of χ, the new (U, V_r) will still satisfy (3.6); however, in general they will not have the special form needed to yield a solution to the m.K.d.V. equations, for example U will not usually be of the form (3.2). We are thus led to the problem: given one acceptable function Ψ, how can we choose χ so that the new (U, V_r) are still of the right kind, i.e. essentially, so that U is of the form (3.2) and V_r is a polynomial in z of degree $2r+1$ with leading term

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} z^{2r+1}?$$

This problem has a simple solution in terms of the Birkhoff (or Riemann–Hilbert) factorization in the group G; we must digress to explain what that is.

Let us first go back to the Lie algebra $\mathfrak{g}$. It has an obvious direct sum decomposition

$$\mathfrak{g} = \mathfrak{u}_- \oplus \mathfrak{h} \oplus \mathfrak{u}_+,$$

where $\mathfrak{u}_-$ (resp. $\mathfrak{u}_+$) is the subalgebra of maps $S^1 \to \mathfrak{sl}(2, \mathbb{C})$ whose Fourier expansion involves only negative (resp. positive) powers of z, and $\mathfrak{h}$ is the one-dimensional subalgebra of constant maps $z \to \operatorname{diag}(a, -a)$, $a \in \mathbb{C}$. Let U_-, H, U_+ be the corresponding subgroups of G; thus in particular H is the group of constant maps $z \to \operatorname{diag}(a, a^{-1})$, $a \in \mathbb{C}^\times$. It is not quite true that the group G decomposes as the product $U_- H U_+$; however, it is true that multiplication defines an injective map from $U_- \times H \times U_+$ onto a dense open subset of G, called the *big cell*. If g belongs to the big cell, we shall write

$$g = g_- g_0 g_+$$

for its unique factorization with $g_- \in U_-, g_0 \in H, g_+ \in U_+$.

Proposition 3. *Let $\Psi(x, \mathbf{t}; z)$ be a G-valued function of $(x, \mathbf{t})$, which is of the right kind for the m.K.d.V. equations (i.e. such that the corresponding U and V_r are of the special form described above). Then for any $g \in G$, the function*

$$g \circ \Psi = (\Psi g \Psi^{-1})_-^{-1} \Psi \tag{3.8}$$

is again of the right kind.

The proof is trivial: if we first calculate the new (U, V_r) from (3.8) we see that they have the right leading terms, but appear to involve some negative powers of z in their Fourier expansions. However, if we do the calculation using the equivalent expression

$$g \circ \Psi = (\Psi g \Psi^{-1})_0 (\Psi g \Psi^{-1})_+ \Psi g^{-1}, \tag{3.9}$$

we see that these negative powers of z are absent.

Proposition 4. *The formula* (3.8) *defines an action of G on the space of suitable functions Ψ; i.e. we have*

$$g \circ h \circ \Psi = (gh) \circ \Psi.$$

We should remark that the last two propositions have been formulated rather imprecisely; for example, we have not specified what class of functions Ψ we want to allow. A more serious point is that we have ignored the fact that the big cell is not the whole of G, that is, that the factorization in the formula (3.8) may not exist. Happily, this problem is not as serious as it might seem: we are saved by the fact that Ψ is not a fixed element of G, but a function of x (and $\mathbf{t}$). One can show that it is not possible for the expression $\Psi g \Psi^{-1}$ to lie outside the big cell for a whole interval of values of x. Thus the formula (3.8) always makes sense provided that we are prepared to allow Ψ to have some singularities: of course this means that the corresponding solutions to the m.K.d.V. equations will have singularities too.

The G-orbit of the trivial solution $q \equiv 0$ to the m.K.d.V. equations is particularly interesting. For the corresponding function Ψ we make the obvious choice

$$\Psi^{(0)} = \exp\left\{(xz + t_3 z^3 + \dots)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}.$$

Since $\Psi^{(0)}$ takes values in the group U_+, the formula (3.8) simplifies slightly: we have

$$g \circ \Psi^{(0)} = (\Psi^{(0)} g)^{-1} \Psi^{(0)}.$$

It is easy to see that $g \circ \Psi^{(0)} = h \circ \Psi^{(0)}$ if and only if g and h belong to the same coset of the group $HU_+ = B$; note that B is just the subgroup of all elements of G, which (in the principal realization) extend holomorphically to the disc D_0. It follows that the G-orbit of $\Psi^{(0)}$ can be identified with the homogeneous space G/B, so that proposition 3 defines a map

$$G/B \to \{\text{solutions of the m.K.d.V. equations}\}.$$

On the other hand, in §2 we assigned a solution of the K.d.V. equations to each point of a different homogeneous space G/P. The connection between these two constructions is very simple. Recall that P is the subgroup of those elements of G that extend holomorphically to the disc D_0 when we use the *standard* realization of G. Now, from (3.4) we easily see that B is a little smaller than this; in fact in the standard realization of G, B consists of those maps $f: S^1 \to \mathrm{SL}(2, \mathbb{C})$ that have a Fourier expansion

$$f(z) = \sum_0^{\infty} a_i z^i$$

in which a_0 is upper triangular. Thus there is a natural projection $G/B \to G/P$, and the fibre P/B can be identified with the Riemann sphere S^2.

Let $\psi(x, \mathbf{t}; z)$ be the scalar valued function obtained by adding the two elements in the first row of the matrix $g \circ \Psi^{(0)}$. Reversing the calculation at the beginning of this section, we see that if q is the solution of the m.K.d.V. equations corresponding to the point $gB \in G/B$, and if we set $\mathrm{L} = (\partial - q)(\partial + q)$, then we have

$$\mathrm{L}\psi = z^2\psi.$$

Proposition 5. *The above function ψ is the Baker function of the point $gP \in G/P \cong Gr^{(2)}$. Thus we have a commutative diagram*

$$\begin{array}{ccc} G/B & \to & \{\text{solutions of the m.K.d.V. equations}\} \\ \downarrow & & \downarrow \\ G/P & \to & \{\text{solutions of the K.d.V. equations}\} \end{array}$$

in which the upper and lower horizontal arrows are those defined in the present section and in §2, *respectively, the left vertical arrow is the natural projection, and the right vertical arrow is the Miura transformation* (1.3).

It may be worth remarking that there is a nice concrete realization of G/B as a flag manifold, corresponding to the realization of G/P as the Grassmannian $Gr^{(2)}$. Indeed, going back to the action of G on the Hilbert space H defined in §2, it is easy to see that $B = P \cap P'$, where P' is the isotropy group of the subspace zH_+ of H. It follows that G/B can be identified with the space of 'periodic flags'

$$z^2 W_0 \subset zW_1 \subset W_0,$$

where each W_i belongs to $Gr^{(2)}$. In this realization, the projection $G/B \to G/P$ is given simply by $(W_0, W_1) \mapsto W_0$.

Finally, we record a slightly more explicit expression for our solutions to the m.K.d.V. equations. It is obtained simply by calculating the x-derivative of $g \circ \Psi^{(0)}$ using the alternative expression (3.9).

Proposition 6. *The solution of the m.K.d.V. equations assigned above to the point $gB \in G/B$ is given by*

$$q(x, \mathbf{t}) = -\partial/\partial x \log (\Psi^{(0)} g)_0. \tag{3.10}$$

Here, as usual in this subject, $\partial/\partial x \log f$ is used as an abbreviation of f_x/f. Of course $(\Psi^{(0)}g)_0$ is really a function of $(x, \mathbf{t})$ with values in H, but we are identifying H with $\mathbb{C}^\times$ in the obvious way (the map $z \to \operatorname{diag}(a, a^{-1})$ is identified with the number a).

4. The group $\hat{G}$ and the τ-functions

The main purpose of this section is to explain why it is natural to rewrite the formula in proposition 6 in the form (1.4), involving certain entire functions τ_i. The essential ingredient is a central extension $\hat{G}$ of the loop group G. Unfortunately this is a less elementary object than any we have encountered so far, and I shall have to omit the details of its construction. However, at the Lie algebra level the situation is quite simple, so we begin with that.

Let $\mathfrak{g}$ be the Lie algebra introduced in §3. Set

$$e_0 = \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix};$$

$$f_0 = \begin{pmatrix} 0 & z^{-1} \\ 0 & 0 \end{pmatrix}, \quad f_1 = \begin{pmatrix} 0 & 0 \\ z^{-1} & 0 \end{pmatrix}.$$

These are all elements of $\mathfrak{g}$, and it is not hard to see that e_0 and e_1 generate the subalgebra $\mathfrak{u}_+$ (topologically), while f_0 and f_1 generate $\mathfrak{u}_-$. The four elements $\{e_i, f_i\}$ together generate the whole of $\mathfrak{g}$, with certain relations that are well known, but which we shall not write down. Set

$$h_i = [e_i, f_i], \quad i = 0, 1,$$

so that

$$h_1 = -h_0 = \operatorname{diag}(1, -1).$$

Thus in $\mathfrak{g}$ we have $h_0 + h_1 = 0$. However, this relation is not a consequence of the other relations in $\mathfrak{g}$, so we can consider the Lie algebra $\hat{\mathfrak{g}}$ defined by the same generators and relations as $\mathfrak{g}$, except that we omit the relation $h_0 + h_1 = 0$, so that h_0 and h_1 are linearly independent elements of $\hat{\mathfrak{g}}$. Then $\hat{\mathfrak{g}}$ is a central extension of $\mathfrak{g}$ with one-dimensional centre $\mathbb{C}$ generated by $h_0 + h_1$. The Lie algebra $\hat{\mathfrak{g}}$ (or more precisely the dense subalgebra of it generated algebraically by the $\{e_i, f_i\}$) is an example of an affine Kac–Moody algebra: it is usually denoted by the symbol $A_1^{(1)}$. Corresponding to the three-part splitting of $\mathfrak{g}$ that we used in §3, we have

$$\hat{\mathfrak{g}} = \mathfrak{u}_- \oplus \hat{\mathfrak{h}} \oplus \mathfrak{u}_+,$$

where $\hat{\mathfrak{h}}$ is the two-dimensional Abelian subalgebra spanned by h_0 and h_1. Note that if we use the bases $\{h_0, h_1\}$ and $\{h_1\}$ to identify $\hat{\mathfrak{h}}$ and $\mathfrak{h}$ with $\mathbb{C}^2$ and $\mathbb{C}$, respectively, then the projection $\pi\colon \hat{\mathfrak{h}} \rightarrow \mathfrak{h}$ is given by

$$\pi(u, v) = v - u.$$

Now, in the infinite-dimensional case it is by no means true that there is a Lie group corresponding to every Lie algebra. Nevertheless, there is a Lie group $\hat{G}$ corresponding to the Lie algebra $\hat{\mathfrak{g}}$: it is a central extension of G by $\mathbb{C}^\times$, and has all the properties we would expect from a consideration of the Lie algebras. The most concrete available construction of $\hat{G}$ is due to G. Segal, and goes along the following lines (for details see Segal & Wilson (1985) or Pressley & Segal (1985)). Go back to the Grassmannian $Gr^{(2)}$ of §2. Over it we have the tautological bundle whose fibre over a point $W \in Gr^{(2)}$ is W itself. By using the theory of Fredholm determinants, it is not hard to make sense of forming a holomorphic line bundle that can be thought of as the 'top exterior power' or determinant bundle of this. The desired group $\hat{G}$ can then be obtained as the group of holomorphic automorphisms of this line bundle: it can be described quite explicitly as a quotient of a certain group of operators on the Hilbert space H.

We now suppose that, by the above method or otherwise, we have constructed the group $\hat{G}$ and proved that it has all the expected properties. In particular, $\hat{G}$ has subgroups $U_-, \hat{H}, U_+$ corresponding to the subalgebras $\mathfrak{u}_-$, $\hat{\mathfrak{h}}$, $\mathfrak{u}_+$, and there is the dense open subset $U_- \hat{H} U_+$ of $\hat{G}$ covering the big cell in G. We use the same notation as before

$$\hat{g} = \hat{g}_- \hat{g}_0 \hat{g}_+$$

for the factorization of an element of $U_- \hat{H} U_+$. Corresponding to our identification of $\hat{\mathfrak{h}}$ and $\mathfrak{h}$ with $\mathbb{C}^2$ and $\mathbb{C}$, the groups $\hat{H}$ and H get identified with $\mathbb{C}^\times \times \mathbb{C}^\times$ and $\mathbb{C}^\times$, respectively; and the projection $\pi\colon \hat{H} \rightarrow H$ is given by

$$\pi(x, y) = x^{-1} y. \tag{4.1}$$

Now fix $g \in G$, and let $q(x, \boldsymbol{t})$ be the solution to the m.K.d.V. equations associated in §3 to the point $gB \in G/B$. Let $\hat{g}$ be any element of $\hat{G}$ with $\pi(\hat{g}) = g$. Since $\Psi^{(0)}(x, \boldsymbol{t}; z)$ takes values in U_+, which is a subgroup of $\hat{G}$, the expression $(\Psi^{(0)} \hat{g})_0$ makes sense: it is a function of $(x, \boldsymbol{t})$

with values in $\hat{H} \cong \mathbb{C}^\times \times \mathbb{C}^\times$. We denote its two components by $\tau_0(x, \boldsymbol{t}), \tau_1(x, \boldsymbol{t})$. Combining the formulas (3.10) and (4.1), we find that the m.K.d.V. solution q is indeed given by

$$q = \partial/\partial x \log \tau_0/\tau_1. \tag{4.2}$$

Now, each τ_i is a function of $(x, \boldsymbol{t})$ with values in $\mathbb{C}^\times$; it is defined only for those values of $(x, \boldsymbol{t})$ such that $\Psi^{(0)}(x, \boldsymbol{t}; z)\, \hat{g}$ belongs to the big cell in $\hat{G}$. Thus *a priori* it looks as if the functions τ_i will have singularities at points $(x, \boldsymbol{t})$ for which this is not so. However, the formula (4.2) is to be considered an advance over (3.10) precisely because that does not happen.

PROPOSITION 7. *The functions τ_0, τ_1 are entire functions of (any number of) the variables $x, t_3, t_5, \ldots$.*

Clearly, this proposition is not of a formal nature, but reflects the fact that $\hat{G}$ is in some sense a 'better' group than G. It is easy to prove the proposition if we suppose that we have constructed holomorphic representations of $\hat{G}$ corresponding to the well known highest weight representations of the affine algebra $A_1^{(1)}$ (see Kac 1983). In particular, let $\mathscr{E}$ be the space of the 'basic' irreducible holomorphic representation of $\hat{G}$; in $\mathscr{E}$ we have the highest weight vector Ω, characterized by the properties that

(i) the subgroup U_+ fixes Ω,

(ii) the subgroup $\hat{H} \cong \mathbb{C}^\times \times \mathbb{C}^\times$ acts on Ω by

$$(x, y)\, \Omega = x\Omega.$$

The space $\mathscr{E}$ has a Hermitian inner product $\langle\ ,\ \rangle$ (we take it to be complex linear in the second variable), which is contravariant with respect to a certain anti-holomorphic involution ω of $\hat{G}$. At the Lie algebra level, ω can be characterized by the properties $\omega(e_i) = -f_i$, where $\{e_i, f_i\}$ are the canonical generators for $\hat{\mathfrak{g}}$ introduced in §3: concerning ω, we need to know only that it interchanges the subgroups U_+ and U_-. 'Contravariant' means that we have

$$\langle u, \hat{g}v\rangle = \langle \omega(\hat{g})\, u, v\rangle.$$

Granting all this, we can make the following simple calculation: for brevity set $\varphi = \Psi^{(0)}\hat{g}$; then

$$\begin{aligned}\langle \Omega, \varphi\Omega\rangle &= \langle \Omega, \varphi_- \varphi_0 \varphi_+ \Omega\rangle \\ &= \langle \omega(\varphi_-)\, \Omega, \varphi_0 \varphi_+ \Omega\rangle \\ &= \langle \Omega, \varphi_0 \Omega\rangle \\ &= \tau_0,\end{aligned}$$

where in the last step we suppose that Ω is normalized so that $\langle \Omega, \Omega\rangle = 1$. We thus have the formula (due to Date *et al.* (1981))

$$\tau_0(x, \boldsymbol{t}) = \langle \Omega, \Psi^{(0)}(x, \boldsymbol{t}; z)\, \hat{g}\Omega\rangle,$$

from which it is clear that τ_0 is an entire function. The argument for τ_1 is the same, by using the other fundamental representation of $\hat{G}$ (characterized as above, but with x and y interchanged in property (ii) of the vector Ω).

We remark that the use of representation theory in the proof of proposition 7 is not essential: in Segal & Wilson (1985) a proof is given that is formulated entirely in terms of the geometry of the determinant line bundle over $Gr^{(2)}$. However, this proof is morally equivalent to the preceding one: the connection is that the space $\mathscr{E}$ of the basic representation can be realized, Borel–Weil fashion, as the space of holomorphic sections of the dual of the determinant bundle.

If we apply the Miura transformation (1.3) to the formula (4.2), then, in view of proposition

5, we get a formula for the solution u of the K.d.V. equations associated to a point gP of G/P. This formula involves both the τ-functions τ_i. However, as I mentioned in the introduction, there is a famous formula for u involving only τ_0, namely

$$u = 2\partial^2/\partial x^2 \log \tau_0.$$

I shall not prove this formula here (it is a surprisingly complicated story), but I do want to explain why, from our present point of view, we should expect u to depend only on τ_0.

We return to the basic representation $\mathscr{E}$ of $\hat{G}$. We can define a map

$$\varphi : \mathscr{E} \rightarrow \{\text{functions of } x, t_3, \ldots\}$$

as follows:

$$\varphi(v)\,(x, t_3, \ldots) = \langle \boldsymbol{\Omega}, \boldsymbol{\Psi}^{(0)}(x, \boldsymbol{t}; z)\, v\rangle.$$

A remarkable fact is that this map is injective, so that the basic representation of $\hat{G}$ can be realized on a suitable space of functions of the (infinitely many) variables $x, t_3, t_5, \ldots$. This is called the *principal realization* of the representation (cf. Kac 1983). If $v = \hat{g}\Omega$ belongs to the orbit of the highest weight vector, then $\varphi(v)$ is just our function τ_0. Now, in the projective space of $\mathscr{E}$, the orbit of the highest weight vector is $G/P \cong Gr^{(2)}$; thus the injectivity of φ implies that a point of $Gr^{(2)}$ is uniquely determined by the corresponding τ-function τ_0. Since the K.d.V. solution u is constructed by starting just from a point of $Gr^{(2)}$, we expect there to be a formula expressing u in terms of τ_0.

As I mentioned in the introduction, when the point $gP \in Gr^{(2)}$ arises from a Riemann surface in the way described in §2, the function τ_0 can be written down explicitly in terms of the Riemann θ-function; for the degenerate cases giving the soliton or rational solutions, τ_0 can be written down in terms of elementary functions. For details I refer to Segal & Wilson (1985); here I just mention the beautiful observation of M. Sato that for the rational solutions certain of the functions τ_0 are Schur polynomials. It would be interesting to have more information about the general τ-functions, corresponding to points of $Gr^{(2)}$ that do not arise from an algebraic curve. At present it seems that nothing is known about them except that they are entire functions.

To conclude, I want to comment on the one part of the above theory that does *not* generalize when we replace G by the loops on any simple Lie group, or, slightly more generally, when G is the loop group corresponding to any affine Kac–Moody algebra $\hat{\mathfrak{g}}$. If l is the rank of $\hat{\mathfrak{g}}$, then $\hat{\mathfrak{g}}$ has $l+1$ fundamental highest weight representations $\mathscr{E}_i$; the orbit of the highest weight vector in the projective space of $\mathscr{E}_i$ is a homogeneous space G/P_i of G, which we can think of as a generalized Grassmannian. Corresponding to each point of G/P_i we have a τ-function τ_i: it is a function on the positive part (in the sense of the principal grading) of the principal Heisenberg subalgebra of $\hat{\mathfrak{g}}$. For each $i = 0, 1, \ldots, l$, Drinfel'd & Sokolov (1981, 1984) have defined a generalized K.d.V. hierarchy; each of these is a quotient of a single m.K.d.V. hierarchy associated with $\hat{\mathfrak{g}}$, via a generalized Miura transformation. The arguments given above apply at once to give a simple formula like (1.4) for a class of solutions of the generalized m.K.d.V. equations in terms of the $l+1$ functions τ_i. Further, for each i we can define a map φ, just as above, from $\mathscr{E}_i$ to the space of functions of infinitely many variables $(x, \boldsymbol{t})$. However, this map is usually not injective, so that in particular a point of a 'Grassmannian' G/P_i is in general not determined by the corresponding τ-function. That is why we can not hope, in general, to have formula like (1.5), expressing solutions of an equation of K.d.V. type in terms of a single τ-function.

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